Mustapha El Hamdaoui [a], José Merodio [a], Ray W. Ogden [b,*], Javier Rodríguez [a,1]

ABSTRACT

We consider the finite radially symmetric deformation of a circular cylindrical tube of a homogeneous transversely isotropic elastic material subject to axial stretch, radial deformation and torsion, supported by axial load, internal pressure and end moment. Two different directions of transverse isotropy are considered: the radial direction and an arbitrary direction in planes normal locally to the radial direction, the only directions for which the considered deformation is admissible in general. In the absence of body forces, formulas are obtained for the internal pressure, and the resultant axial load and torsional moment on the ends of the tube in respect of a general strain-energy function. For a specific material model of transversely isotropic elasticity, and material and geometrical parameters, numerical results are used to illustrate the dependence of the pressure, (reduced) axial load and moment on the radial stretch and a measure of the torsional deformation for a fixed value of the axial stretch.

1. Introduction

It is well known from the literature (see, for example, the review by Saccomandi (2001)) that, in the absence of body forces, a number of deformations can be supported in equilibrium in an incompressible isotropic nonlinearly elastic solid material by application of surface tractions alone. Such deformations are said to be *controllable*. If, within a given class of materials, the deformation is controllable for all materials and independent of any specific constitutive law in the considered class then the deformation is said to be *universal* (within the considered class). Saccomandi (2001) introduced the term *relative-universal* for situations where the class of materials is a subclass of a general class of materials. The deformation that is of particular interest in the present paper is a combined deformation consisting of the (i) finite extension, (ii) inflation and (iii) torsion of a cylindrical circular tube, which, for an isotropic material, is indeed universal. However, for anisotropic materials, in particular for the transversely isotropic materials with which we are concerned is this paper, this deformation is only controllable for certain directions of transverse isotropy, and then, in these cases, it is also universal.

Several authors have studied the deformations (i)–(iii) for isotropic materials from many different perspectives in the past. In brief, torsional deformations for incompressible isotropic materials were first examined in a series of papers by Rivlin (1948, 1949a,b) while associated experimental data were provided in Rivlin (1947) and Rivlin and Saunders (1951). Gent and Rivlin (1952), guided by the theoretical results of Rivlin (1949b), performed experiments to obtain data for the problem of combined uniform extension, uniform inflation and small amplitude torsion. Comparison of the Ogden model (Ogden, 1972) for rubberlike solids with the data given in Rivlin and Saunders (1951) for solid and tubular cylinders composed of natural rubber under combined extension and torsional deformation has been presented by Ogden and Chadwick (1972). A detailed analysis of the combined extension and inflation of such materials with particular reference to bifurcation into non-circular cylindrical modes of deformation was provided by Haughton and Ogden (1979a,b).

More recently, Horgan and Saccomandi (1999) used a material model incorporating limiting chain extensibility to capture the hardening response of incompressible isotropic elastic materials under large strain torsional deformations, while Kanner and Horgan (2008) were concerned with investigating the effects of strain stiffening on the response of solid circular cylinders in the combined deformation of torsion superimposed on axial extension.

For compressible isotropic materials, for which the deformations (i)–(iii) are not, in general, controllable, a class of materials admitting isochoric pure torsional deformation was proposed by Polignone and Horgan (1991). In the same spirit, Kirkinis and Ogden (2002) derived analogous solutions and also introduced a methodology for generating corresponding results for incompressible materials. Different aspects of pure torsion for special classes of compressible materials and considerations of loss of ellipticity have also been studied in Beatty (1996) and Horgan and Polignone (1995), respectively, amongst others.

The contributions mentioned above related to rubberlike materials, but more recently attention has also been focused on elastic deformations of soft biological tissues in the context of biomechanics, and these materials are in general anisotropic, typically transversely isotropic or orthotropic. To the best of our knowledge, very few authors have studied the deformations (i)–(iii) for *anisotropic* elastic solids in the finite deformation regime, in particular for incompressible transversely isotropic elastic solids, including fibre-reinforced materials, although Green and Adkins (1970) presented some general theoretical results for a transversely isotropic circular cylindrical tube subject to axial extension, inflation and torsion for the case in which the axis of transverse isotropy is aligned with the tube axis. Also, under the restriction of idealized fibre reinforcement (i.e. inextensible fibres), Spencer (1972) discussed the problem of extension and torsion of solid elastic cylinders augmented with one or two families of helical fibres, although the analysis is mainly restricted to the linear theory (see also the interesting discussion relating to two symmetric helically disposed fibre families in Spencer (1984)). For large deformations, in the context of soft tissue biomechanics (with particular reference to arteries), the problem of extension and inflation has been examined by Ogden and Schulze-Bauer (2000), with the anisotropy associated with helical fibre reinforcement, which is used to model the contribution of embedded collagen fibres to the overall response of the tissue, while Horgan and Saccomandi (2003) discussed the combined extension and inflation problem for soft tissues by taking into account limiting chain extensibility. A thorough analysis of the elastic response of arteries, for simultaneous extension, inflation and torsion, was provided by Holzapfel et al. (2000).

In the present analysis, we consider the problem of combined finite extension/contraction, radial contraction/expansion and torsion of a circular cylindrical tube of homogeneous elastic material with specific directions of transverse isotropy (which may be, but need not necessarily be considered as a material reinforced by a single family of fibres). In particular, in Section 2 we introduce the notation and summarize the necessary kinematics for the combined deformation in an incompressible material. We then summarize, in Section 3, the constitutive equation for a transversely isotropic material, and the equilibrium equations (in the absence of body forces) are used to obtain general formulas for the internal pressure in the tube, the resultant axial load and moment on the ends of the tube that are applied to maintain the prescribed deformation in respect of a general transversely isotropic form of constitutive law. These results, which also apply in the isotropic specialization, recover the formulas given in Haughton and Ogden (1979a), for the case in which no torsion is applied to the tube.

In Section 4 we highlight the fact that, for transversely isotropic materials, the considered deformation cannot be maintained for all possible directions of transverse isotropy, and we therefore specialize to those directions which are admissible, specifically the radial direction and directions locally lying in planes normal to the radius of the tube.

In general, closed form solutions are not obtainable in simple form, and in order to illustrate the results we therefore provide numerical results based on a simple prototype form of transversely isotropic strain-energy function in Section 5. In particular, we show, in graphical form, how, for a fixed value of the axial extension, the pressure, the (reduced) axial load and the moment depend on the applied torsion and radial stretch for a specific tube thickness and transverse isotropy parameter. Finally, a brief summary of the results is given in the concluding Section 6.

2. Kinematics and geometry

Consider a material continuum which, when unstressed and unstrained, occupies the *reference* configuration $\mathcal{B}_r$. Let a typical material point in this configuration be identified by its position vector $\mathbf{X}$. The corresponding position vector in the deformed configuration $\mathcal{B}$ is denoted $\mathbf{x}$ and the deformation from $\mathcal{B}_r$ to $\mathcal{B}$ is written $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$, where the vector function $\boldsymbol{\chi}$ is referred to as the *deformation* (we are considering quasi-static deformations here). The deformation gradient tensor, denoted $\mathbf{F}$, is given by

$$\mathbf{F} = \mathrm{Grad}\,\boldsymbol{\chi}(\mathbf{X}), \tag{1}$$

where Grad is the gradient operator with respect to $\mathbf{X}$. The associated right and left Cauchy–Green deformation tensors, denoted $\mathbf{C}$ and $\mathbf{B}$ respectively, are defined as

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{U}^2, \quad \mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}} = \mathbf{V}^2, \tag{2}$$

where $\mathbf{U}$ and $\mathbf{V}$, respectively, are the right and the left stretch tensors, which are positive definite and symmetric and come from the polar decompositions $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$, $\mathbf{R}$ being a proper orthogonal tensor. For a homogeneous incompressible nonlinearly isotropic elastic solid, the elastic stored energy (defined per unit volume) depends on only two invariants, which are the principal invariants of $\mathbf{C}$ (equivalently of $\mathbf{B}$), defined by

$$I_1 = \mathrm{tr}(\mathbf{C}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \mathrm{tr}(\mathbf{C}^{-1}) = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2, \tag{3}$$

where $\lambda_i > 0, i \in \{1,2,3\}$, are the principal stretches, i.e. the eigenvalues of $\mathbf{U}$ and $\mathbf{V}$. The incompressibility constraint, which in terms of $\mathbf{F}$ is

$$\det \mathbf{F} = 1, \tag{4}$$

may be written in terms of the principal stretches as

$$\lambda_1\lambda_2\lambda_3 = 1. \tag{5}$$

If the material has a single distinguished direction (the direction of transverse isotropy), identified by the unit vector $\mathbf{M}$ in the reference configuration, two more invariants, denoted I_4 and I_5 (in general independent), are introduced that are associated with $\mathbf{M}$. These invariants are defined by

$$I_4 = \mathbf{FM}\cdot\mathbf{FM} = \mathbf{m}\cdot\mathbf{m}, \quad I_5 = \mathbf{CM}\cdot\mathbf{CM} = \mathbf{m}\cdot\mathbf{Bm}, \tag{6}$$

where we have introduced the vector $\mathbf{m} = \mathbf{FM}$, which represents the direction of transverse isotropy in the deformed configuration. In general $\mathbf{m}$ is not a unit vector.

2.1. Combined extension, inflation and torsion

We now consider a circular cylindrical tube, which, in terms of cylindrical polar coordinate (R, Θ, Z), is defined by

$$A \leqslant R \leqslant B, \quad 0 \leqslant \Theta \leqslant 2\pi, \quad 0 \leqslant Z \leqslant L \tag{7}$$

in the reference configuration $\mathcal{B}_r$, where A and B are the internal and external radii and L is the length of the tube. The position vector $\mathbf{X}$ of a point of the tube is given by

$$\mathbf{X} = R\mathbf{E}_R + Z\mathbf{E}_Z, \tag{8}$$

where $\mathbf{E}_R$ and $\mathbf{E}_Z$ are the unit basis vectors associated with R and Z, respectively. We also denote by $\mathbf{E}_\Theta$ the corresponding unit vector associated with Θ.

The position vector $\mathbf{x}$ in the deformed tube is written

$$\mathbf{x} = r\mathbf{e}_r + z\mathbf{e}_z, \tag{9}$$

where we make use of cylindrical polar coordinates (r, θ, z) in $\mathcal{B}$, which are associated with unit basis vectors $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$. The (isochoric) deformation consisting of axial extension, radial inflation and a superimposed torsion is defined by

$$r = \sqrt{a^2 + \lambda_z^{-1}(R^2 - A^2)}, \quad \theta = \Theta + \tau\lambda_z Z, \quad z = \lambda_z Z, \tag{10}$$

where λ_z is the (uniform) axial stretch of the cylinder, τ is the torsional deformation per unit deformed length (plane cross sections of the tube remain plane and an initial radius at station Z turns through an angle τz after axial extension), and the deformed geometry of the tube is defined by

$$a \leqslant r \leqslant b, \quad 0 \leqslant \theta \leqslant 2\pi, \quad 0 \leqslant z \leqslant l = \lambda_z L. \tag{11}$$

For this deformation the deformation gradient is calculated explicitly as

$$\mathbf{F} = \lambda_r \mathbf{e}_r \otimes \mathbf{E}_R + \lambda_\theta \mathbf{e}_\theta \otimes \mathbf{E}_\Theta + \lambda_z \mathbf{e}_z \otimes \mathbf{E}_Z + \lambda_z \gamma \mathbf{e}_\theta \otimes \mathbf{E}_Z, \tag{12}$$

where we have defined γ as $\gamma = \tau r$ and λ_r and λ_θ are the principal stretches in the radial and azimuthal directions *prior* to application of the torsion. Once the torsion is applied λ_θ and λ_z are no longer principal stretches. Nevertheless, the incompressibility constraint (5) becomes

$$\lambda_r \lambda_\theta \lambda_z = 1, \tag{13}$$

which is independent of γ. In general, application of the torsion will change the geometry given by (11) but here we fix the length l during torsion and the internal radius a and ensure that the circular cylindrical configuration is maintained. The deformation tensors (2) are calculated as

$$\begin{aligned}
\mathbf{C} &= \lambda_r^2 \mathbf{E}_R \otimes \mathbf{E}_R + \lambda_\theta^2 \mathbf{E}_\Theta \otimes \mathbf{E}_\Theta + \lambda_z^2(1+\gamma^2)\mathbf{E}_Z \otimes \mathbf{E}_Z + \gamma\lambda_z\lambda_\theta(\mathbf{E}_\Theta \otimes \mathbf{E}_Z + \mathbf{E}_Z \otimes \mathbf{E}_\Theta),\\
\mathbf{B} &= \lambda_r^2 \mathbf{e}_r \otimes \mathbf{e}_r + (\lambda_\theta^2 + \gamma^2\lambda_z^2)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \lambda_z^2 \mathbf{e}_z \otimes \mathbf{e}_z + \gamma\lambda_z^2(\mathbf{e}_\theta \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_\theta).
\end{aligned} \tag{14}$$

For the considered deformation one of the Eulerian principal axes remains aligned with $\mathbf{e}_r$ and corresponds to the principal stretch $\lambda_1 = \lambda_r$. Now let $\mathbf{v}^{(i)}, i \in \{1,2,3\}$, be the unit Eulerian principal axes associated with the deformation (i.e. the principal axes of $\mathbf{B}$), and $\lambda_i, i \in \{1,2,3\}$, be the corresponding principal stretches. Then, we may express $\mathbf{v}^{(i)}$ in terms of $\mathbf{e}_r, \mathbf{e}_\theta$ and $\mathbf{e}_z$, in the form

$$\begin{aligned}
&\mathbf{v}^{(1)} = \mathbf{e}_r, \quad \mathbf{v}^{(2)} = \cos\psi\,\mathbf{e}_\theta + \sin\psi\,\mathbf{e}_z,\\
&\quad\mathbf{v}^{(3)} = -\sin\psi\,\mathbf{e}_\theta + \cos\psi\,\mathbf{e}_z,
\end{aligned} \tag{15}$$

where ψ identifies the orientation of the axes $\mathbf{v}^{(2)}$ and $\mathbf{v}^{(3)}$ in the $(\mathbf{e}_\theta, \mathbf{e}_z)$ plane.

The spectral decomposition of $\mathbf{B}$ is given by

$$\mathbf{B} = \sum_{i=1}^{3} \lambda_i^2 \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)} \tag{16}$$

and combining this with $(14)_2$ and using (15) we obtain the connections

$$\begin{aligned}
&\lambda_1 = \lambda_r, \quad (\lambda_2^2 - \lambda_3^2)\sin\psi\cos\psi = \gamma\lambda_z^2,\\
&\lambda_2^2\cos^2\psi + \lambda_3^2\sin^2\psi = \lambda_\theta^2 + \gamma^2\lambda_z^2, \quad \lambda_2^2\sin^2\psi + \lambda_3^2\cos^2\psi = \lambda_z^2,
\end{aligned} \tag{17}$$

from which we deduce that

$$\begin{aligned}
&\lambda_2\lambda_3 = \lambda_\theta\lambda_z, \quad \lambda_2^2 + \lambda_3^2 = \lambda_\theta^2 + \lambda_z^2(\gamma^2 + 1),\\
&\quad(\lambda_2^2 - \lambda_3^2)\cos 2\psi = \lambda_\theta^2 + \lambda_z^2(\gamma^2 - 1), \quad (\lambda_2^2 - \lambda_3^2)\sin 2\psi = 2\gamma\lambda_z^2
\end{aligned} \tag{18}$$

and then

$$\tan 2\psi = \frac{2\gamma\lambda_z^2}{\lambda_\theta^2 + \lambda_z^2(\gamma^2 - 1)}. \tag{19}$$

Note that in different notation the above equations were derived by Ogden and Chadwick (1972); see also Kirkinis and Ogden (2002) for the case of extension and torsion of a compressible elastic circular cylinder.

It is also easily shown that λ_2 and λ_3 are given in terms of $\lambda_\theta, \lambda_z$ and γ through

$$\begin{aligned}
\lambda_2^2 &= \lambda_\theta^2\cos^2\psi + \lambda_z^2(\gamma\cos\psi + \sin\psi)^2,\\
\lambda_3^2 &= \lambda_\theta^2\sin^2\psi + \lambda_z^2(\gamma\sin\psi - \cos\psi)^2.
\end{aligned} \tag{20}$$

Since $\lambda_\theta, \lambda_z$ and γ are independent, the properties

$$\begin{aligned}
&\lambda_\theta\frac{\partial\lambda_2}{\partial\lambda_\theta} + \lambda_z\frac{\partial\lambda_2}{\partial\lambda_z} = \lambda_2, \quad \lambda_\theta\frac{\partial\lambda_3}{\partial\lambda_\theta} + \lambda_z\frac{\partial\lambda_3}{\partial\lambda_z} = \lambda_3,\\
&\lambda_\theta\frac{\partial\lambda_2}{\partial\lambda_\theta} + \gamma\frac{\partial\lambda_2}{\partial\gamma} = \lambda_2\cos^2\psi, \quad \lambda_\theta\frac{\partial\lambda_3}{\partial\lambda_\theta} + \gamma\frac{\partial\lambda_3}{\partial\gamma} = \lambda_3\sin^2\psi
\end{aligned} \tag{21}$$

can be established from (18).

As a prelude to the next section we record here the explicit expressions for the invariants I_1, I_2, I_4, I_5 for the considered deformation with the direction $\mathbf{M}$ having components (M_R, M_Θ, M_Z) with respect to the reference cylindrical polar coordinates:

$$\begin{aligned}
I_1 &= \lambda_r^2 + \lambda_\theta^2 + \lambda_z^2(1+\gamma^2),\\
I_2 &= \lambda_\theta^2\lambda_z^2 + \lambda_r^2\lambda_z^2(1+\gamma^2) + \lambda_r^2\lambda_\theta^2,\\
I_4 &= \lambda_r^2 M_R^2 + (\lambda_\theta M_\Theta + \gamma\lambda_z M_Z)^2 + \lambda_z^2 M_Z^2,\\
I_5 &= \lambda_r^4 M_R^2 + \lambda_\theta^2(\lambda_\theta^2 + \gamma^2\lambda_z^2)M_\Theta^2 + 2\gamma\lambda_\theta\lambda_z(\lambda_\theta^2 + \lambda_z^2 + \gamma^2\lambda_z^2)M_\Theta M_Z\\
&\quad + \lambda_z^2[\gamma^2\lambda_\theta^2 + (1+\gamma^2)^2\lambda_z^2]M_Z^2.
\end{aligned} \tag{22}$$

3. Constitutive laws

For a homogeneous incompressible elastic solid the strain energy is a function only of the deformation gradient $\mathbf{F}$, and we write the strain-energy function as $W(\mathbf{F})$ per unit volume, although, by objectivity, W depends on $\mathbf{F}$ only through the right Cauchy–Green tensor $\mathbf{C}$ defined in (2). The Cauchy stress tensor $\boldsymbol{\sigma}$ is given by

$$\boldsymbol{\sigma} = \mathbf{F}\frac{\partial W}{\partial \mathbf{F}} - p\mathbf{I}, \tag{23}$$

where p is a Lagrange multiplier associated with the incompressibility constraint (4) and $\mathbf{I}$ is the identity tensor.

For a transversely isotropic material, W depends on the invariants I_1, I_2, I_4, I_5, and we write $W = W(I_1, I_2, I_4, I_5)$ without changing the notation for the functional dependence of W. The Cauchy stress can then be expanded out in the standard form

$$\begin{aligned}
\boldsymbol{\sigma} = {}& 2W_1\mathbf{B} + 2W_2(I_1\mathbf{I} - \mathbf{B})\mathbf{B} + 2W_4\mathbf{m} \otimes \mathbf{m} + 2W_5(\mathbf{m} \otimes \mathbf{Bm}\\
&+ \mathbf{Bm} \otimes \mathbf{m}) - p\mathbf{I},
\end{aligned} \tag{24}$$

where $W_i = \partial W/\partial I_i$, $i \in \{1,2,4,5\}$. The reference configuration $\mathcal{B}_r$, in which $I_1 = I_2 = 3$ and $I_4 = I_5 = 1$, is taken to be stress free, and the strain energy W is measured from the reference configuration. Thus, on evaluation of (24) in $\mathcal{B}_r$,

$$\begin{aligned}
&2W_1(3,3,1,1) + 4W_2(3,3,1,1) = p_0,\\
&\quad W_4(3,3,1,1) + 2W_5(3,3,1,1) = 0,
\end{aligned} \tag{25}$$

as well as $W(3,3,1,1) = 0$, as given by Merodio and Ogden (2002), where p_0 is the value of p in the reference configuration.

For the considered deformation the invariants I_1, I_2, I_4, I_5 depend on only three independent deformation variables, which we take to be $\lambda_\theta, \lambda_z$ and γ, while λ_r is given by (13) in terms of λ_θ and λ_z, and we write the strain energy as a function of these three variables, specifically $\hat{W}(\lambda_\theta, \lambda_z, \gamma)$, which is defined by

$$\hat{W}(\lambda_\theta, \lambda_z, \gamma) = W(I_1, I_2, I_4, I_5), \tag{26}$$

with I_1, I_2, I_4, I_5 given by (22).

Then, a straightforward calculation using the appropriate specialization of the components of the Cauchy stress in (24) leads to the compact formulas

$$\sigma_{\theta\theta} - \sigma_{rr} = \lambda_\theta \frac{\partial \hat{W}}{\partial \lambda_\theta} + \gamma \frac{\partial \hat{W}}{\partial \gamma}, \quad \sigma_{\theta z} = \frac{\partial \hat{W}}{\partial \gamma},$$

$$\sigma_{\theta\theta} + \sigma_{zz} - 2\sigma_{rr} = \lambda_\theta \frac{\partial \hat{W}}{\partial \lambda_\theta} + \lambda_z \frac{\partial \hat{W}}{\partial \lambda_z}. \tag{27}$$

We emphasize that these hold independently of the direction of the vector **M**. Note, however, that no corresponding simple formulas are available for the stress components $\sigma_{r\theta}$ and σ_{rz}. An equivalent set of equations, expressed in terms of the components of the Green strain tensor $\mathbf{E} = (\mathbf{C} - \mathbf{I})/2$, was given by Holzapfel et al. (2000).

3.1. Equilibrium and boundary loads

In the absence of body forces the Cauchy stress tensor must satisfy the equilibrium equation $\mathrm{div}\boldsymbol{\sigma} = \mathbf{0}$. Since λ_θ and γ depend on one spatial variable only, namely r, and λ_z is uniform, the cylindrical polar components of Cauchy stress likewise depend only on r, and the equilibrium equation therefore reduces to the three standard component equations

$$r\frac{\mathrm{d}}{\mathrm{d}r}(\sigma_{rr}) + \sigma_{rr} - \sigma_{\theta\theta} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}r}(r^2\sigma_{r\theta}) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}r}(r\sigma_{rz}) = 0. \tag{28}$$

Eq. $(28)_1$ can be integrated to give

$$\sigma_{rr}(b) - \sigma_{rr}(a) = \int_a^b (\sigma_{\theta\theta} - \sigma_{rr})\frac{\mathrm{d}r}{r}, \tag{29}$$

where $\sigma_{rr}(a)$ and $\sigma_{rr}(b)$ are the values of the radial stress σ_{rr} on the boundaries $r = a$ and $r = b$, respectively. Eqs. $(28)_{2,3}$ are integrated immediately to give

$$\sigma_{r\theta} = \frac{c_1}{r^2}, \quad \sigma_{rz} = \frac{c_2}{r}, \tag{30}$$

where c_1 and c_2 are constants. In general, because the deformation is known as a function of r, i.e. $\gamma = \tau r, \lambda_\theta \equiv r/R = r/\sqrt{[\lambda_z(r^2 - a^2) + A^2]}$, and λ_z is constant, the latter two solutions are untenable since they are not compatible with the specific forms of $\sigma_{r\theta}$ and σ_{rz} that arise from the constitutive equation, as will become clear in Section 4, and we shall consider only the special situations in which they are possible solutions. This means that we can only consider situations in which the constitutive law leads to $\sigma_{r\theta} = \sigma_{rz} = 0$, and hence $c_1 = c_2 = 0$.

Let us now suppose that the curved boundaries of the tube are subject to an inflating pressure P on $r = a$ and no radial traction on $r = b$, so that $\sigma_{rr}(a) = -P$ and $\sigma_{rr}(b) = 0$. Then, on use of $(27)_1$, Eq. (29) becomes

$$P = \int_a^b \left(\lambda_\theta \frac{\partial \hat{W}}{\partial \lambda_\theta} + \gamma \frac{\partial \hat{W}}{\partial \gamma}\right)\frac{\mathrm{d}r}{r}. \tag{31}$$

The resultant axial load N on an end of the tube (and on any cross section) is calculated from

$$N = \int_a^b \int_0^{2\pi} \sigma_{zz} r\,\mathrm{d}r\,\mathrm{d}\theta = 2\pi \int_a^b \sigma_{zz} r\,\mathrm{d}r, \tag{32}$$

which can be rearranged in a standard way using $(28)_1$ and the boundary values of σ_{rr} to obtain

$$N = \pi \int_a^b (2\sigma_{zz} - \sigma_{rr} - \sigma_{\theta\theta}) r\,\mathrm{d}r + \pi a^2 P \tag{33}$$

and hence, on use of (27),

$$N = \pi \int_a^b \left(2\lambda_z \frac{\partial \hat{W}}{\partial \lambda_z} - \lambda_\theta \frac{\partial \hat{W}}{\partial \lambda_\theta} - 3\gamma \frac{\partial \hat{W}}{\partial \gamma}\right) r\,\mathrm{d}r + \pi a^2 P. \tag{34}$$

A corresponding expression is obtained for the resultant moment M. This is defined as

$$M = \int_a^b \int_0^{2\pi} \sigma_{\theta z} r^2\,\mathrm{d}r\,\mathrm{d}\theta \tag{35}$$

and hence, on use of $(27)_2$,

$$M = 2\pi \int_a^b \frac{\partial \hat{W}}{\partial \gamma} r^2\,\mathrm{d}r. \tag{36}$$

Note that, since $\gamma = \tau r$, Eq. (36) can be used to rewrite (34) as

$$N = \pi \int_a^b \left(2\lambda_z \frac{\partial \hat{W}}{\partial \lambda_z} - \lambda_\theta \frac{\partial \hat{W}}{\partial \lambda_\theta}\right) r\,\mathrm{d}r - \frac{3}{2}\tau M + \pi a^2 P. \tag{37}$$

We also make use of the so-called *reduced axial load* N_r defined by

$$N_\mathrm{r} = N - \pi a^2 P = \pi \int_a^b \left(2\lambda_z \frac{\partial \hat{W}}{\partial \lambda_z} - \lambda_\theta \frac{\partial \hat{W}}{\partial \lambda_\theta}\right) r\,\mathrm{d}r - \frac{3}{2}\tau M, \tag{38}$$

which reduces the total axial load by the axial contribution due to the pressure on the ends of a closed-ended tube.

4. Admissible directions of transverse isotropy

As indicated above, the solutions (30) are not in general compatible with the expressions for $\sigma_{r\theta}$ and σ_{rz} obtained from (24), which are

$$\sigma_{r\theta} = 2\lambda_r M_R\{\lambda_\theta M_\Theta[W_4 + (\lambda_r^2 + \lambda_\theta^2 + \gamma^2\lambda_z^2)W_5] + \gamma\lambda_z M_Z[W_4 + (\lambda_r^2 + \lambda_\theta^2 + \gamma^2\lambda_z^2 + \lambda_z^2)W_5]\}, \tag{39}$$

$$\sigma_{rz} = 2\lambda_r M_R\{\lambda_\theta\gamma\lambda_z^2 M_\Theta W_5 + \lambda_z M_Z[W_4 + (\lambda_r^2 + \gamma^2\lambda_z^2 + \lambda_z^2)W_5]\}. \tag{40}$$

For the problem of extension, inflation and torsion non-zero stresses $\sigma_{r\theta}$ and σ_{rz} are generated by the presence of the preferred direction, except for the situations in which either $M_R = 0$ or $M_\Theta = M_Z = 0$ when they both vanish and $c_1 = c_2 = 0$, as is also the case for an isotropic material ($\mathbf{M} = \mathbf{0}$). If we take $M_Z = 0$, for example, they reduce to

$$\sigma_{r\theta} = 2\lambda_z^{-1} M_R M_\Theta[W_4 + (\lambda_r^2 + \lambda_\theta^2 + \gamma^2\lambda_z^2)W_5], \quad \sigma_{rz} = 2\gamma\lambda_z M_R M_\Theta W_5, \tag{41}$$

but still the solutions (30) are not in general consistent with these expressions. Similarly if we take $M_\Theta = 0$ instead of $M_Z = 0$.

In the following, therefore, we consider only the two cases identified above, i.e. $M_R = 1$ and $M_R = 0$. For these two cases the considered deformation is controllable and also relative-universal in the full class of transversely isotropic elastic materials, for each case separately.

4.1. Radial transverse isotropy

First we consider the case of transverse isotropy in the radial direction, so $M_R = 1$ and $M_\Theta = M_Z = 0, \mathbf{m} = \mathbf{FM} = \lambda_r\mathbf{e}_r, I_4 = \lambda_r^2$ and $I_5 = I_4^2$. In particular, we see that under extension and inflation ($\lambda_\theta > 1$ and $\lambda_z > 1$) the radial direction is compressed. Here we are allowing the direction of transverse isotropy to support compression, which is commonly not the case when associating the direction with that of biopolymer fibres or filaments such as collagen (Holzapfel et al., 2000). Note, in particular, that neither I_4 nor I_5 depends on γ, and thus the presence of the transverse isotropy does not contribute more to the strain energy as a result of torsion than the corresponding isotropic material. Moreover, we have $\sigma_{r\theta} = \sigma_{rz} = 0$ from (39) and (40), so that $c_1 = c_2 = 0$.

In fact, it is easy to see that in this case $\boldsymbol{\sigma}$ is coaxial with **B**, just as for an isotropic material, and therefore yields the universal relation

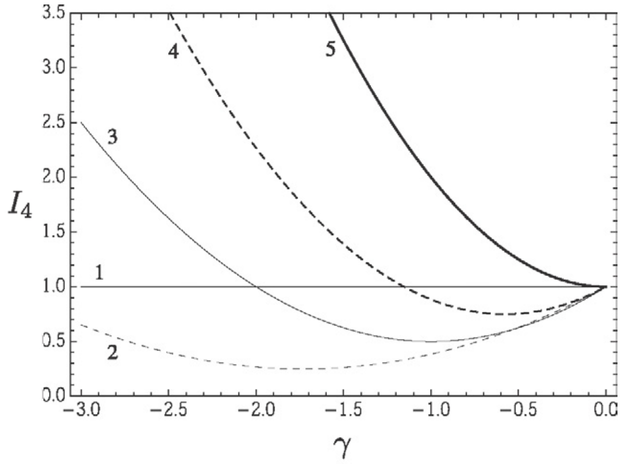

Fig. 1. Plot of I_4 against negative γ for $\lambda_\theta = \lambda_z = 1$ with $\alpha = 0, \pi/6, \pi/4, \pi/3, \pi/2$, labelled 1, 2, 3, 4, 5, respectively.

$$\frac{\gamma\lambda_z^2}{\lambda_\theta^2 + \gamma^2\lambda_z^2 - \lambda_z^2} = \frac{\sigma_{\theta z}}{\sigma_{\theta\theta} - \sigma_{zz}} \quad \left(= \frac{1}{2}\tan 2\psi\right). \tag{42}$$

For general discussion of universal relations we refer to Pucci and Saccomandi (1997) and Saccomandi (2001).

4.2. Transverse isotropy with $M_R = 0$

First we note that the case $M_R = 0$ includes the case of circumferential transverse isotropy, for which $M_\Theta = 1, M_Z = 0, \mathbf{m} = \mathbf{FM} = \lambda_\theta \mathbf{e}_\theta$, and hence $I_4 = \lambda_\theta^2$ and $I_5 = I_4^2$. Again the invariants are independent of γ, but, unlike the radial case, $\boldsymbol{\sigma}$ is not coaxial with $\mathbf{B}$. More generally, when the direction of transverse isotropy is not circumferential but has no radial component $\mathbf{M}$ can be written

$$\mathbf{M} = \cos\alpha \mathbf{E}_\Theta + \sin\alpha \mathbf{E}_Z, \tag{43}$$

where α, with $0 \leqslant \alpha \leqslant \pi/2$, is the angle that the direction makes locally with the azimuthal direction. In this situation the invariants I_4 and I_5 are given by

$$\begin{aligned} I_4 &= (\lambda_\theta \cos\alpha + \gamma\lambda_z \sin\alpha)^2 + \lambda_z^2 \sin^2\alpha, \\ I_5 &= (\lambda_\theta \cos\alpha + \gamma\lambda_z \sin\alpha)^2 (\lambda_\theta^2 + \gamma^2\lambda_z^2) + \lambda_z^4 \sin^2\alpha \\ &\quad + 2\gamma\lambda_z^3(\lambda_\theta \cos\alpha + \gamma\lambda_z \sin\alpha)\sin\alpha \end{aligned} \tag{44}$$

and it can be shown that

$$I_5 = I_4(I_1 - \lambda_\theta^{-2}\lambda_z^{-2}) - \lambda_\theta^2\lambda_z^2. \tag{45}$$

The stress components $\sigma_{r\theta}$ and σ_{rz} are both zero in this situation, and the remaining stress components are given by (27), with $M_R = 0$ being implicit.

For positive γ, both I_4 and I_5 are monotonic increasing functions of each of γ, λ_θ and λ_z. Thus, in this case the preferred directions are extending. For $\gamma < 0$ and λ_z fixed, I_4 has a minimum when $\gamma\lambda_\theta/\lambda_z = -\cot\alpha$, and its minimum value may be expressed as $\lambda_z^2 \sin^2\alpha$, which may be greater than or less than 1. To illustrate this we plot, in Fig. 1, I_4 against γ for negative values of γ with $\lambda_\theta = \lambda_z = 1$. For other values of λ_θ and λ_z the plots are similar. As pointed out in Merodio and Ogden (2005) for the simple shear of fibre-reinforced materials, this means that certain preferred directions contract as γ decreases from zero, reach a minimum length after which the length increases until it surpasses its initial value and continues to extend.

The situation with I_5 is not so straightforward, and closed form results for a minimum are not forthcoming, but for certain values of α two minima can be found, as shown in Merodio and Ogden (2005) for simple shear. Because of this complication we do not consider I_5 further in this paper.

5. A neo-Hookean transversely isotropic material

In order to illustrate the results further we consider a specific form of strain-energy function, denoted $\overline{W}$ and defined by

$$\overline{W} = \frac{\mu}{2}\left[I_1 - 3 + \rho(I_4 - 1)^2\right], \tag{46}$$

where μ (>0) is a constant, which would correspond to the shear modulus in the undeformed configuration of a neo-Hookean isotropic material, while ρ (>0) is a constant that measures the strength of the reinforcement. The first term corresponds to the energy of the neo-Hookean base material and the second term to the energy associated with the direction of transverse isotropy, referred to as the *standard reinforcing model* in situations where the isotropic base material is reinforced with fibres of a stronger material.

We apply this model to the specific cases of the transversely isotropic directions mentioned in the previous section. In each case we give results for P, M and the reduced axial load $N_r = N - \pi a^2 P$ in dimensionless form P^*, M^* and N_r^*, defined by

$$P^* = \frac{P}{\mu}, \quad M^* = \frac{M}{\pi\mu A^3}, \quad N_r^* = \frac{N_r}{\pi\mu A^2} \tag{47}$$

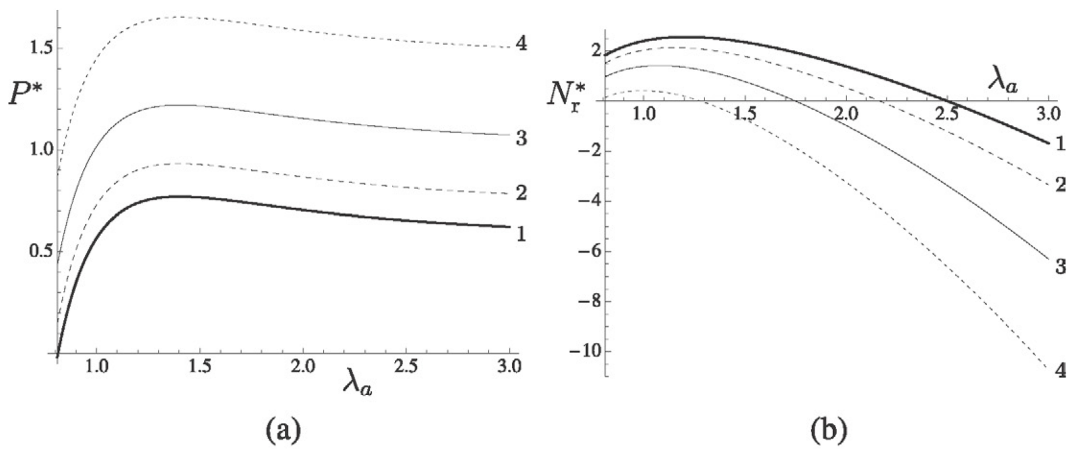

Fig. 2. Plots of (a) the dimensionless pressure $P^* = P/\mu$ and (b) the reduced axial load $N_r^* = N_r/(\pi A^2\mu)$ against λ_a for the case of radial transverse isotropy with $\rho = 4, \eta = 4, \lambda_z = 1.2$ and the following values of the dimensionless torsional strain τ^* : 0, 0.3, 0.5, 0.7, labelled 1, 2, 3, 4, respectively.

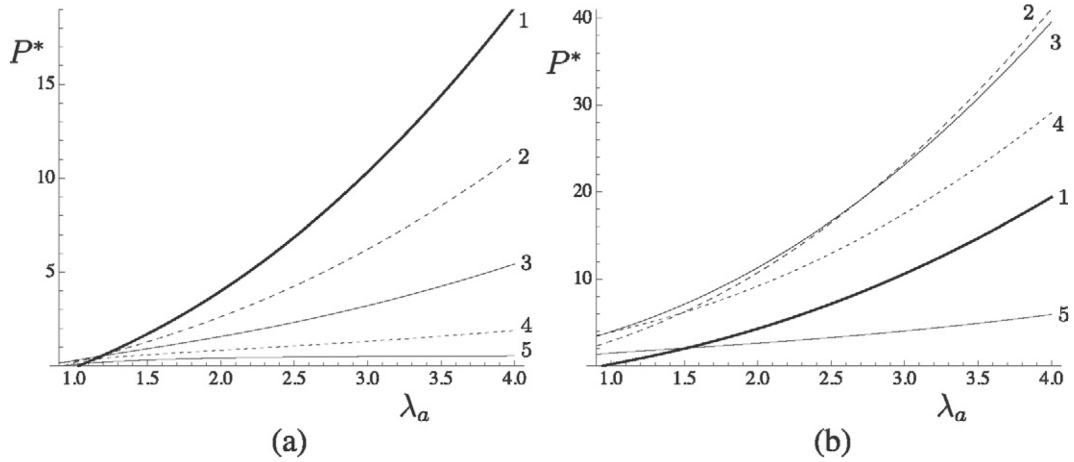

Fig. 3. Plots of the dimensionless pressure P^* versus λ_a for $\rho = 4, \eta = 4, \lambda_z = 1.2$ with $\alpha = 0, \pi/6, \pi/4, \pi/3, \pi/2$ (curves labelled 1, 2, 3, 4, 5, respectively): (a) $\tau^* = 0$; (b) $\tau^* = 0.4$.

and we define the notations $\eta = B^2/A^2, \tau^* = \tau A$. In general, the angle α is allowed to depend on R but for simplicity here we take it to be constant.

5.1. Radial transverse isotropy

In this case the preferred direction is represented by the unit vector $\mathbf{M}$ with $M_R = 1$, and it is straightforward to show that

$$M^* = \frac{1}{2}\tau^*(\eta - 1)(2\lambda_a^2\lambda_z + \eta - 1), \tag{48}$$

which depends linearly on both τ and λ_z and quadratically on λ_a and η, where $\lambda_a = a/A$ is the value of λ_θ on $r = a$. Note, in particular, that M^* does not depend on the transverse isotropy. This is not surprising since at fixed λ_θ and λ_z the torsion just rotates the direction of transverse isotropy and does not change its length.

Formulas for P^* and N_r^* can also be obtained explicitly, but their expressions are quite lengthy and not enlightening, so we omit them and just provide some numerical results. In particular, for the fixed values of $\rho = 4, \eta = 4$ and $\lambda_z = 1.2$ we plot them in Fig. 2 versus λ_a for a series of values of τ^*, specifically $\tau^* = 0, 0.3, 0.5, 0.7$.

Suppose that the axial stretch λ_z is applied first with $P = 0$. This requires an axial load N and, for non-zero τ^*, end moments M. The value of λ_a at $P = 0$ depends on both λ_z and τ^*, and then the pressure increases as λ_a is increased until it reaches a maximum, after which it decreases monotonically to an approximately constant value, as illustrated in Fig. 2(a) for several different values of τ^*. For a given value of λ_a, larger values of τ^* require larger values of P^*. On the other hand, in Fig. 2(b) the curves show that the dimensionless

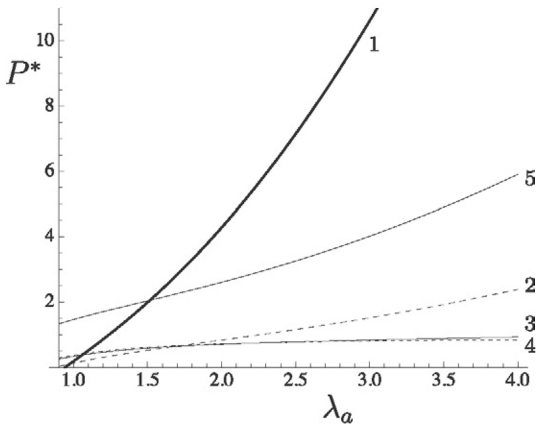

Fig. 4. Plots of the dimensionless pressure P^* versus λ_a for $\rho = 4, \eta = 4, \lambda_z = 1.2$ with $\alpha = 0, \pi/6, \pi/4, \pi/3, \pi/2$ (curves labelled 1, 2, 3, 4, 5, respectively): $\tau^* = -0.4$.

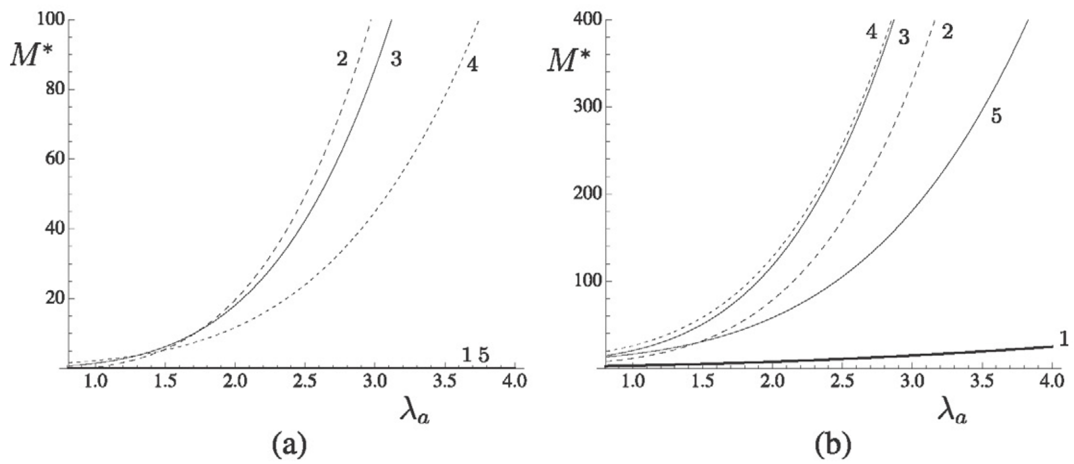

Fig. 5. Plots of the dimensionless moment M^* versus λ_a for $\rho = 4, \eta = 4, \lambda_z = 1.2$ with $\alpha = 0, \pi/6, \pi/4, \pi/3, \pi/2$ (curves labelled 1, 2, 3, 4, 5, respectively): (a) $\tau^* = 0$; (b) $\tau^* = 0.4$. In (a), for $\alpha = 0, \pi/2, M^* \equiv 0$.

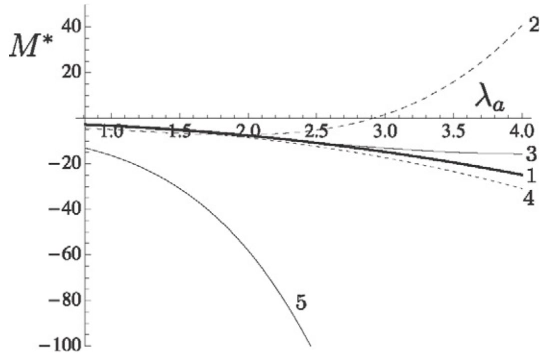

Fig. 6. Plots of the dimensionless moment M^* versus λ_a for $\rho = 4, \eta = 4, \lambda_z = 1.2$ with $\alpha = 0, \pi/6, \pi/4, \pi/3, \pi/2$ (curves labelled 1, 2, 3, 4, 5, respectively): $\tau^* = -0.4$.

reduced axial load N_r^*, which is positive initially, increases up to a maximum, after which it decreases as λ_a increases and then becomes negative, increasingly so for larger τ^*.

5.2. Transverse isotropy with $M_R = 0$

In this case $\mathbf{M} = M_\Theta \mathbf{E}_\Theta + M_Z \mathbf{E}_Z$, where $M_\Theta = \cos\alpha$ and $M_Z = \sin\alpha$. For the special case of circumferential transverse isotropy ($\alpha = 0$) M^* is again given by (48), but the expressions for P^* and N_r^* are quite lengthy even for this case and even more so for the general case, so are not included here. The results for P^* and N_r^* are again illustrated numerically, along with results for M^*, as functions of λ_a for the values $\rho = 4, \eta = 4, \lambda_z = 1.2$ as well as for different values of τ^* and α.

Again the axial stretch is applied first, requiring not only an axial load but also, depending on α, a moment, even when $\tau^* = 0$. Thus, the value of λ_a at $P = 0$ depends on λ_z, τ^* and α, and P is then increased from zero as λ_a is increased from its value at $P = 0$.

First, Figs. 3 and 4 show the results for P^*. The qualitative behaviour of each of the curves in Fig. 3(a), for which $\tau^* = 0$, is the same, and all the curves increase monotonically with λ_a. Moreover, for a given value of λ_a larger values of α are associated with smaller values of the pressure. For the curve 5, for which $\alpha = \pi/2$, it follows from $(44)_1$ that $I_4 = \lambda_z^2$ and the radial behaviour of the tube is just that of a neo-Hookean isotropic material (the term in ρ in (46) does not contribute to the pressure).

When τ^* is not zero the clear pattern of Fig. 3(a) is significantly altered, as illustrated in Fig. 3(b) for positive τ^* and Fig. 4 for neg-

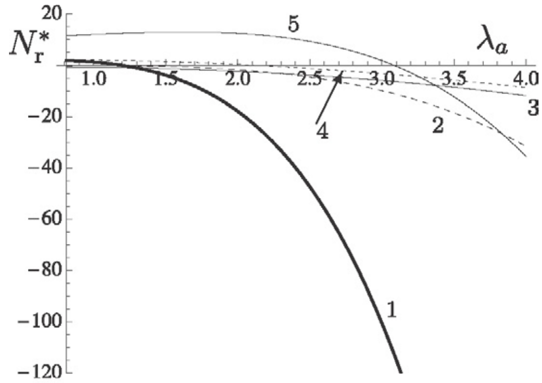

Fig. 8. Plots of the dimensionless reduced axial load N_r^* versus λ_a for $\rho = 4, \eta = 4, \lambda_z = 1.2$ with $\alpha = 0, \pi/6, \pi/4, \pi/3, \pi/2$ (curves labelled 1, 2, 3, 4, 5, respectively): $\tau^* = -0.4$.

ative τ^*. Indeed, the results for positive and negative τ^* are very different because the directions of transverse isotropy have different influences in the two cases. For example, for $\tau^* = 0.4$ the curve 2 for $\alpha = \pi/6$ or 3 for $\alpha = \pi/4$, requires the largest value of the pressure to achieve a given radial expansion beyond a certain level whereas it is curve 1 for $\tau^* = 0$ or for $\tau^* = -0.4$. It is worth remarking that for a thin-walled tube, but not in the case of a thick-walled tube, it is easy to show, by approximating the expression (31) and using the energy function (46), that for given values of τ, λ_z and the initial radius R of the tube, there is an angle α for which the pressure is independent of ρ. This is given by $\tan\alpha = -1/(\tau R \lambda_z)$, and in this case the radial response of the tube is again neo-Hookean in character. Note that the curve 1 in Fig. 4 is the same as that in Fig. 3(a) and (b), although this is not apparent since the vertical scales are different. We also note that the results for negative τ^* can be reproduced from those for positive τ^* by changing the range of values of the angle α.

In Figs. 5 and 6, plots of M^* against λ_a are provided, corresponding to those in Figs. 3 and 4, respectively, for the three values of τ^*. It is clear that for $\tau^* = 0$, by symmetry, the moment is exactly zero for $\alpha = 0$ and $\alpha = \pi/2$, as shown in Fig. 5(a). Furthermore, for $\alpha = 0$ it follows that $I_4 = \lambda_\theta^2$ (independent of γ) and the value of M^*, as given by (48), corresponds to that for the neo-Hookean case. Again, there is a sharp distinction in the behaviour for positive and negative values of τ^*. In Fig. 5(b), corresponding to positive τ^*, the moment is positive for all angles α, but, for negative τ^*, as Fig. 6 illustrates, M^*, although mainly negative, can be positive for cer-

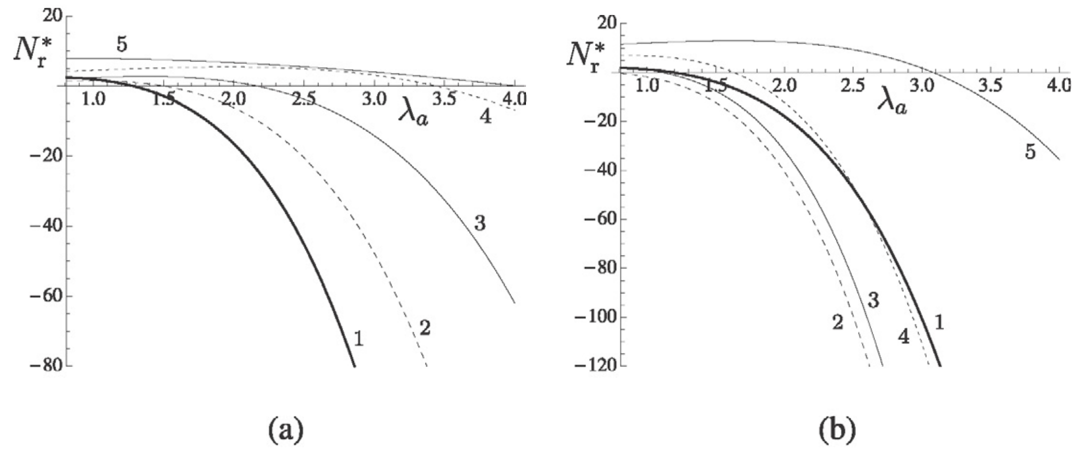

Fig. 7. Plots of the dimensionless reduced axial load N_r^* versus λ_a for $\rho = 4, \eta = 4, \lambda_z = 1.2$ with $\alpha = 0, \pi/6, \pi/4, \pi/3, \pi/2$ (curves labelled 1, 2, 3, 4, 5, respectively): (a) $\tau^* = 0$; (b) $\tau^* = 0.4$.

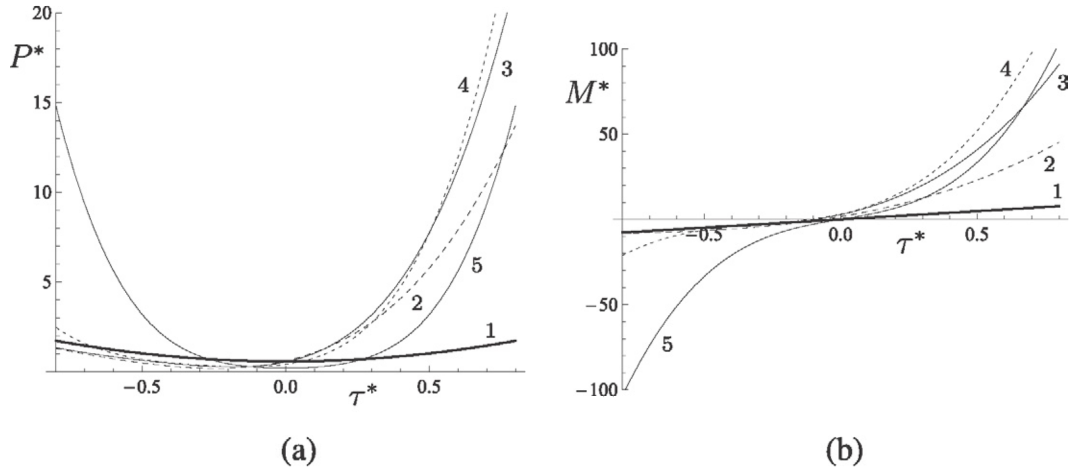

Fig. 9. Plot of (a) the dimensionless pressure P^* and (b) the moment M^* versus τ^* for $\rho = 4, \eta = 4$ and stretches $\lambda_z = 1.2$ and $\lambda_a = 1.2$ and for $\alpha = 0, \pi/6, \pi/4, \pi/3, \pi/2$ (curves labelled $1, 2, 3, 4, 5$, respectively).

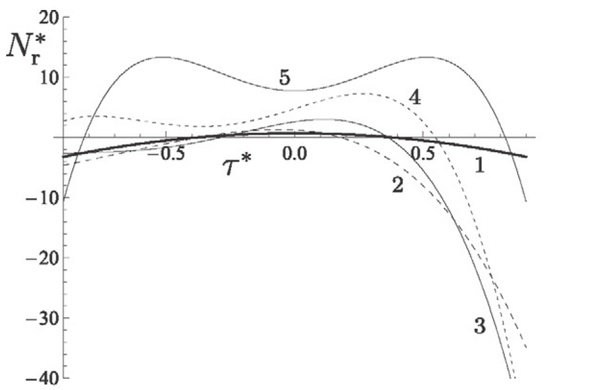

Fig. 10. Plot of the dimensionless reduced axial load N_r^* versus τ^* for $\rho = 4, \eta = 4$ and stretches $\lambda_z = 1.2$ and $\lambda_a = 1.2$ and for $\alpha = 0, \pi/6, \pi/4, \pi/3, \pi/2$ (curves labelled $1, 2, 3, 4, 5$, respectively).

tain angles α for sufficiently large radial expansion. Apart from sign the curves 1 in Fig. 5(b) and Fig. 6 are the same.

Figs. 7 and 8 provide plots of N_r^* against λ_a for the three values of τ^*. As can be seen from Fig. 7(a), as the angle α decreases from $\pi/2$ to 0 the reduced axial load turns from positive to negative as λ_a increases. Again, when τ^* is not zero the ordering of the curves for different α changes, and there is a significant difference in the response between positive and negative values of τ^*. The curves 1 are the same in each case, and the curves 5 in Fig. 7(b), and Fig. 8 are also the same.

Thus far, all the plots show curves plotted with λ_a as the abscissa. To provide and alternative view, we now fix $\lambda_a = 1.2$ and $\lambda_z = 1.2$ and plot P^*, M^* and N_r^* in Figs. 9(a) and (b) and 10, respectively, as functions of τ^* for the same representative angles α as in the previous plots. The pressure and the reduced axial force are even functions of τ^* for $\alpha = 0$ and $\alpha = \pi/2$, but not for the other angles. We note that all the $\alpha = 0$ curves correspond in general to smaller magnitudes of P^*, M^* and N_r^* than are associated with the other angles. This is because, for $\alpha = 0, I_4$ does not depend on τ^*. The transverse isotropy therefore has a stiffening effect.

6. Concluding remarks

In this paper we have shown that the considered deformation of extension, inflation and torsion of a thick-walled circular cylindrical tube is only controllable for certain directions of transverse isotropy, and for these directions it is also universal relative to the considered class of transverse isotropy. We have considered a very simple model of transverse isotropy by way of illustration, and have shown, even for this simple model, how the direction of transverse isotropy has a significant effect on the pressure, (reduced) axial load and torsional moment response of a thick-walled elastic tube undergoing finite deformations. When specialized to the linear theory, this model, as shown in Merodio and Ogden (2005), yields only two elastic constants instead of the three associated with a general incompressible transversely isotropic linearly elastic material. More general nonlinear models that do capture the full linear specialization with three elastic constants can of course be considered and will undoubtedly lead to somewhat different results. The present model, however, is sufficient for our purposes.

We should also point out that restricted forms of transversely isotropic constitutive laws are not in general able to fully capture available experimental data. In particular, both I_4 and I_5 are needed if the infinitesimal longitudinal and transverse shear moduli of the material are different, as explained by Murphy (2013); see also the discussions in Destrade et al. (2013) and Pucci and Saccomandi (2014).

Acknowledgement

The authors acknowledge support from the Ministerio de Ciencia, Spain, under the Project number DPI2011-26167. Mustapha El Hamdaoui also thanks the Ministerio de Economía y Competitividad, Spain, for funding under Grant DPI2008-03769.

References

Beatty, M.F., 1996. Introduction to nonlinear elasticity. In: Carroll, M.M., Hayes, M.A. (Eds.), Nonlinear Effects in Fluids and Solids. Plenum Press, New York, pp. 13–112.

Destrade, M., MacDonald, B., Murphy, J.G., Saccomandi, G., 2013. At least three invariants are necessary to model the mechanical response of incompressible, transversely isotropic materials. Comput. Mech 52, 959–969.

Gent, A.N., Rivlin, R.S., 1952. Experiments on the mechanics of rubber II: the torsion, inflation and extension of a tube. Proc. Phys. Soc. B 65, 487–501.

Green, A.E., Adkins, J.E., 1970. Large Elastic Deformations, second ed. Clarendon Press, Oxford.

Haughton, D.M., Ogden, R.W., 1979a. Bifurcation of inflated circular cylinders of elastic material under axial loading I. Membrane theory for thin-walled tubes. J. Mech. Phys. Solids 27, 179–212.

Haughton, D.M., Ogden, R.W., 1979b. Bifurcation of inflated circular cylinders of elastic material under axial loading II. Exact theory for thick-walled tubes. J. Mech. Phys. Solids 27, 489–512.
Holzapfel, G.A., Gasser, T.C., Ogden, R.W., 2000. A new constitutive framework for arterial wall mechanics and a comparative study of material models. J. Elasticity 61, 1–48.
Horgan, C.O., Polignone, D.A., 1995. A note on the pure torsion of a circular cylinder for a compressible nonlinearly elastic material with nonconvex strain-energy. J. Elasticity 37, 167–178.
Horgan, C.O., Saccomandi, G., 1999. Simple torsion of isotropic, hyperelastic, incompressible materials with limiting chain extensibility. J. Elasticity 56, 159–170.
Horgan, C.O., Saccomandi, G., 2003. A description of arterial wall mechanics using limiting chain extensibility constitutive models. Biomech. Model. Mechanobiol. 1, 251–266.
Kanner, L., Horgan, C.O., 2008. On extension and torsion of strain-stiffening rubber-like elastic circular cylinders. J. Elasticity 93, 39–61.
Kirkinis, E., Ogden, R.W., 2002. On extension and torsion of a compressible elastic circular cylinder. Math. Mech. Solids 7, 373–392.
Merodio, J., Ogden, R.W., 2002. Material instabilities in fiber-reinforced nonlinearly elastic solids under plane deformation. Arch. Mech. 54, 525–552.
Merodio, J., Ogden, R.W., 2005. Mechanical response of fiber-reinforced incompressible non-linearly elastic solids. Int. J. Non-Linear Mech. 40, 213–227.
Murphy, J.G., 2013. Transversely isotropic biological, soft tissue must be modelled using both anisotropic invariants. Eur. J. Mech. A/Solids 42, 90–96.
Ogden, R.W., 1972. Large deformation isotropic elasticity – on the correlation of theory and experiment for incompressible rubberlike solids. Proc. R. Soc. Lond. A 326, 565–584.
Ogden, R.W., Chadwick, P., 1972. On the deformation of solid and tubular cylinders of incompressible isotropic elastic materials. J. Mech. Phys. Solids. 20, 77–90.
Ogden, R.W., Schulze-Bauer, C.A.J., 2000. Phenomenological and structural aspects of the mechanical response of arteries. In: Casey, J., Bao, G. (Eds.), Mechanics in Biology, AMD-242/BDE-46, pp. 125–140.
Polignone, D.A., Horgan, C.O., 1991. Pure torsion of compressible non-linearly elastic circular cylinders. Q. Appl. Math. 49, 591–607.
Pucci, E., Saccomandi, G., 1997. On universal relations in continuum mechanics. Contin. Mech. Thermodyn. 9, 61–72.
Pucci, E., Saccomandi, G., 2014. On the use of universal relations in the modeling of transversely isotropic materials. Int. J. Solids Struct. 51, 377–380.
Rivlin, R.S., 1947. Torsion of a rubber cylinder. J. Appl. Phys. 18, 444–449.
Rivlin, R.S., 1948. Large elastic deformations of isotropic materials IV. Further developments of the general theory. Phil. Trans. R. Soc. Lond. A 241, 379–397.
Rivlin, R.S., 1949a. A note on the torsion of an incompressible highly-elastic cylinder. Math. Proc. Cambridge Phil. Soc. 45, 485–487.
Rivlin, R.S., 1949b. Large elastic deformations of isotropic materials VI. Further results in the theory of torsion, shear and flexure. Philos. Trans. R. Soc. Lond. A 242, 173–195.
Rivlin, R.S., Saunders, D.W., 1951. Large elastic deformations of isotropic materials VII. Experiments on the deformation of rubber. Philos. Trans. R. Soc. Lond. A 243, 251–288.
Saccomandi, G., 2001. Universal results in finite elasticity. In: Fu, Y.B., Ogden, R.W. (Eds.), Nonlinear Elasticity. Theory and Applications. Cambridge University Press, pp. 97–134.
Spencer, A.J.M., 1972. Deformations of Fibre-reinforced Materials. Oxford University Press.
Spencer, A.J.M., 1984. Continuum Theory of the Mechanics of Fibre-Reinforced Composites. CISM Courses and Lecture Notes, vol. 282. Springer, Wien, pp. 1–32.